Generating Functions and the Fibonacci Sequence

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1 Introduction

This handout will briefly discuss generating functions and one of their applications. Specifically, we will show how to use them to find closed-form expressions for recursively defined functions, using the Fibonacci Sequence as an example. We will show that

$$F(n) = \frac{1}{\sqrt{5}}(\phi_+^n - \phi_-^n)$$

where $F(n)$ is the n^{th} Fibonacci number and $\phi_\pm := \frac{1\pm\sqrt{5}}{2}$. As a refresher, recall that the Fibonacci numbers are defined by

$$F(n) = F(n-1) + F(n-2)$$

for $n \geq 2$ and $F(0) = 0$, $F(1) = 1$, yielding the famous sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, ...

2 Generating Functions

As a bit of background knowledge:

Definition. Given a sequence $\{a(n)\}_{n\geq 0} \subseteq \mathbb{N}$, we define the generating function for the sequence to be

$$\sum_{n=0}^{\infty} a(n)q^n = a(0) + a(1)q + a(2)q^2 + a(3)q^3 + \cdots$$

where $|q| < 1$ is a formal variable. Notice that this form is similar to the power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

which converges when $|x| < 1$.

Additionally if we let $A(q) = \sum a(n)q^n$ and $B(q) = \sum b(n)q^n$ where $a(n)$ and $b(n)$ are arithmetic functions, we have some useful properties regarding addition and subtraction:

$$A(q) \pm B(q) = \sum_{n=0}^{\infty} a(n)q^n \pm \sum_{n=0}^{\infty} b(n)q^n = \sum_{n=0}^{\infty}[a(n) \pm b(n)]q^n$$

And multiplication:

$$A(q)B(q) = C(q) = \sum_{n=0}^{\infty}, \text{ where } c(n) = \sum_{k=0}^{n} a(n-k)b(n) = \sum_{k=0}^{n} a(n)b(n-k)$$

3 The Fibonacci Numbers

Now let's apply over to the Fibonacci numbers. The generating function for $F(n)$ has the form

$$\begin{aligned}\sum_{n=0}^{\infty} F(n)q^n = q + q^2 + 2q^3 + 3q^4 + 5q^5 + \cdots = \sum_{n=1}^{\infty} F(n)q^n = q + \sum_{n=2}^{\infty}[F(n-1) + F(n-2)]q^n \\ = q + \sum_{n=2}^{\infty} F(n-1)q^n + \sum_{n=2}^{\infty} F(n-2)q^n = q + q\sum_{n=1}^{\infty} F(n)q^n + q^2\sum_{n=0}^{\infty} F(n)q^n \\ = q + q\sum_{n=1}^{\infty} F(n)q^n + q^2\sum_{n=1}^{\infty} F(n)q^n\end{aligned}$$

So we have that

$$\sum_{n=1}^{\infty} F(n)q^n = q + q\sum_{n=1}^{\infty} F(n)q^n + q^2\sum_{n=1}^{\infty} F(n)q^n$$

Which yields

$$\sum_{n=0}^{\infty} F(n)q^n = \sum_{n=1}^{\infty} F(n)q^n = \frac{q}{1-q-q^2}$$

Now the point we made about power series earlier comes in handy. A power series seems like the intuitive direction to head, as generating functions resemble them in some sense and geometric power series provide closed-form expressions for equations. In order to express this generating function as a power series, we need to partially decompose the fraction $\frac{q}{1-q^2-q^2}$. Solving for the roots of the denominator $1-q-q^2$ gives $q = -\phi_{\pm} = \frac{-1\pm\sqrt{5}}{2}$. Then the denominator factors cleanly into $1-q-q^2 = -(x+\phi_+)(x+\phi_-)$. So,

$$\sum_{n=0}^{\infty} F(n)q^n = \frac{q}{1-q-q^2} = \frac{-q}{(q+\phi_-)(q+\phi_+)} = \frac{A}{q+\phi_-} + \frac{B}{q+\phi_-}$$

Solving for A and B, we get that $A = \frac{\phi_-}{\sqrt{5}}$ and $B = \frac{-\phi_+}{\sqrt{5}}$. Our overall expression becomes

$$\sum_{n=0}^{\infty} F(n)q^n = \frac{1}{\sqrt{5}}\left(\frac{\phi_-}{q+\phi_-} - \frac{\phi_+}{q+\phi_+}\right)$$

Notice that $\phi_+ = \frac{-1}{\phi_-}$ So we can then rewrite the terms in our generating function as

$$\frac{\phi_+}{q+\phi_+} = \frac{1}{1+\frac{q}{\phi_+}} = \frac{1}{1-q\phi_-} = \sum_{n=0}^{\infty} \phi_-^n q^n$$

Similarly,

$$\frac{\phi_-}{q+\phi_-} = \sum_{n=0}^{\infty} \phi_+^n q^n$$

Which allows us to rewrite our generating function as

$$\sum_{n=0}^{\infty} F(n)q^n = \frac{1}{\sqrt{5}} \left(\sum_{n=0}^{\infty} \phi_+^n q^n - \sum_{n=0}^{\infty} \phi_-^n q^n \right) = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left(\phi_+^n - \phi_-^n \right) q^n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left(\phi_+^n - \phi_-^n \right) q^n$$

It follows that

$$F(n) = \frac{1}{\sqrt{5}} \left(\phi_+^n - \phi_-^n \right)$$

as desired.

4 The Golden Ratio

Interestingly enough, the Fibonacci Sequence is closely tied to the Golden Ratio, which was discovered in Ancient Greece. Two numbers are in the golden ratio if their ratio is the same as that of their sum to the larger of the two numbers. Expressed algebraically, given a, $b \in \mathbb{R}$, the a and b are in the golden ratio ϕ if

$$\phi = \frac{a}{b} = \frac{b}{a+b} = \frac{1+\sqrt{5}}{2}$$

which is the same value as ϕ_+ from our equation above. In addition, the golden ratio has been alternatively expressed in terms of the never-ending fraction

$$\phi = [1; 1, 1, 1, ...] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

or the progression of square roots

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}$$

Additionally, as the golden ratio is defined by a geometric relation, it has numerous application within geometry, especially within partitioning shapes likes triangles, rectangles, and pentagons.

5 Other Remarks

Aside from this coincidental relation to the Golden Ratio however, the Fibonacci sequence doesn't really seem to be all that special after all. Any recursively defined function can be defined explicitly in a closed-form using generating functions (we can even come up with a random one ourselves and find a similar expression with different terms). It's clear that generating functions are pretty powerful tools in finding alternate ways to represent arithmetic sequences. Being able to express a generating function in different ways can lead to insights about the coefficients of a function, and even help us come up with alternative forms to express it and reveal some of its properties in the process. All in all, generating functions are pretty cool and don't take an insane amount of mathematical background to get your feet wet with them—this proof required just a bit of algebra and some basic understanding of series. In the future, I hope to write on more advanced topics regarding the application of generating functions, including their usage in Fourier Analysis and modular forms.