Brouwer's Fixed Point Theorem and Applications

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In this handout, I will discuss Brouwer's Fixed Point Theorem and provide a proof of its 2-dimension case. The proof encompasses much of general topology and requires background in concepts like metric spaces, compactness, and continuity. At the end I'll discuss some applications of Brouwer's Theorem, one of which motivated this post. Credit to Bruce Hughes for providing the original guideline of the proof.

1 Introduction

Brouwer's Fixed Point Theorem is a famous theorem in topology and has useful applications in differential equations, geometry, and even economics. Conceptually, the theorem is not too difficult to understand and I'll use a couple of examples for motivation. Suppose you're stirring a cup of coffee with a spoon without spilling it–when all the coffee particles come to rest, there's going to be at least one particle of coffee that ended up at exactly the same point in the cup it started in.

Another, more easily illustrated example deals with two maps of the same country. If you have two different maps of the U.S. lying on top of one another, then we have a point on the smaller map that lies directly above the same point on the larger map, aka a fixed point that is constant between the two map.

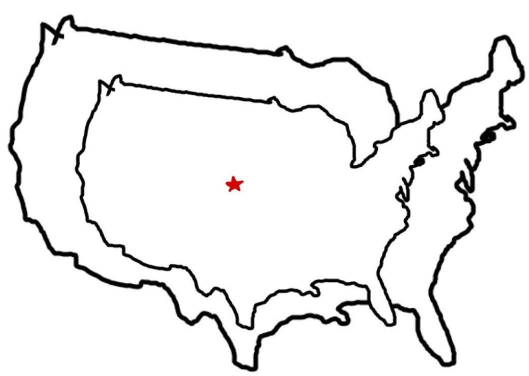


Figure 1: The red star corresponds with the "fixed point"

Now we have a bit to work with regarding fixed points to build up to Brouwer's Theorem. Intuitively, there seems to be this property of some spaces that point(s) will remain fixed after some sort of operation has been done to them. We turn to the next section for a more rigorous discussion.

2 Brouwer's Fixed Point Theorem

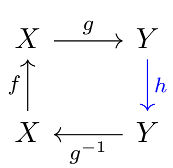
In order to discuss Brouwer's Fixed Point Theorem we have to first talk about fixed points and their properties. Let's begin by recalling a couple of definitions:

> **Definition 2.1** (Fixed Point) — Let X be a topological space and $f : X \to X$ be a continuous function. Then $x \in X$ is a **fixed point** of f means that $f(x) = x$. Furthermore, X has the **fixed point property** (fpp) if and only if every continuous function $f : X \to X$ has a fixed point.

Clearly, not every function is continuous, and not every continuous function has a fixed point. For example, $f(x) = x + 1$ has no fixed points since for no real numbers does $x = x + 1$. So the fixed point property seems to be relatively special as it requires *all* continuous functions within a topological space to have a fixed point. Seeing as our example $f(x) = x + 1$ in $\mathbb{R}$ does not have a fixed point, it follows that $\mathbb{R}$ does not have the fixed point property. In fact, we can actually comment specifically on how the fixed point property is special:

> **Theorem 2.2** — The fixed point property is a topological property. That is if X and Y are homeomorphic topological spaces and X has the fixed point property, then Y also has the fixed point property.

Proof. Let $f : X \to X$ be a self-map and $p \in X$ s.t. $f(p) = p$. As $X \cong Y$, there exists a continuous bijection $g : X \to Y$ whose inverse is also continuous. Define $q := g(p)$; then it follows that $g^{-1}(q) = p$ Then, as can be seen my the commuting diagram, a self-map $h : Y \to Y$ can be reconstructed as as $h = g \circ f \circ g^{-1} : Y \to Y$. Then, $h(q) = g(f(g^{-1}(q))) = g(f(p)) = g(p) = q$. It follows then that any $h : Y \to Y$ has a point $q \in Y$ s.t. $h(q) = q$. Then, Y has the fixed point property as well.



□

Having discussed fixed points, the next natural thing is to formally introduce Brouwer's Fixed Point Theorem. Brouwer's Theorem applies to a subset of Euclidean space called convex sets.

Definition 2.3 (Convex Set) — Let X be a subset of Euclidean space. Then X is a **convex set** means for every $p, q \in X$, the line segment that connects p and q is fully contained in X.

This can be illustrated in the following figure and leads us to Brouwer's Theorem:

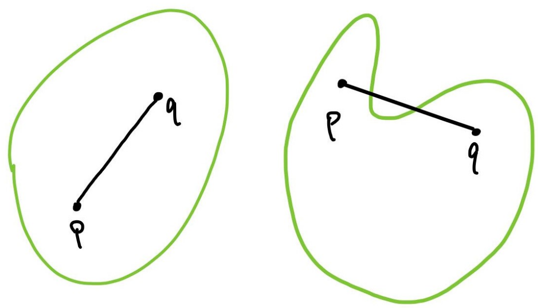


Figure 2: A convex set (left) and a non-convex set (right)

Theorem 2.4 (Brouwer's Fixed Point Theorem) — Let X be a compact convex set and $f : X \to X$ be a continuous function. Then f has a fixed point. That is, every compact convex set has the fixed point property.

Proof. The proof of Brouwer's Theorem in higher dimensional cases (≥ 2) requires advanced concepts in algebraic topology, but the proof for the 2-dimensional case can be done sufficiently without much algebra and focuses mainly on applying general topology. □

Note that the theorem does not say that every compact convex set is homeomorphic to one another–Theorem 2.2 simply stated that homeomorphic spaces share topological properties between one another. This does not mean that spaces that share topological properties imply homeomorphisms.

Moving forward, the dimension-2 case of Brouwer's Theorem focuses on the unit circle or, rather, everything contained in the unit circle in $\mathbb{R}^2$. I choose specifically to focus on the proof in dimension-2 since it serves as a good review of general topology and does not require as much background in fundamental algebra as higher-dimensional analogs.

Theorem 2.5 (Brouwer's Theorem Dim-2) — Let $f : B^2 \to B^2$ be a continuous function where $B^2 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Then f has a fixed point or equivalently, B^2 has the fixed point property.

Proof. The proof of this will be the focus on the next section. Note that B^2 is the set of all points that are encompassed by the unit circle. Another way to represent B^2 is the closed ball about the origin in $\mathbb{R}^2$ with radius 1. $\square$

3 Proof of Dimension 2

4 Applications